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Asymptotic properties of supercritical age-dependent branching processes and homogeneous branching random walks

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Abstract

Let $(Z(t)\colon t \geqslant 0)$ be a supercritical age-dependent branching process and let $\{Y_n\}$ be the natural martingale arising in a homogeneous branching random walk. Let Z be the almost sure limit of $Z(t)/EZ(t)(t \to \infty)$ or that of Y_n $(n \to \infty)$. We study the following problems: (a) the absolute continuity of the distribution of Z and the regularity of the density function; (b) the decay rate (polynomial or exponential) of the left tail probability $P(Z \leqslant x)$ as $x \to 0$, and that of the characteristic function $E\mathrm{e}^{\mathrm{i}tZ}$ and its derivative as $|t| \to \infty$; (c) the moments and decay rate (polynomial or exponential) of the right tail probability $P(Z > x)$ as $x \to \infty$, the analyticity of the characteristic function $\phi(t) = E\mathrm{e}^{\mathrm{i}tZ}$ and its growth rate as an entire characteristic function. The results are established for non-trivial solutions of an associated functional equation, and are therefore also applicable for other limit variables arising in age-dependent branching processes and in homogeneous branching random walks.

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0. Introduction and description of models

An *age-dependent branching process* – the Bellman–Harris process – can be described as follows. A particle existing at time 0 is assumed to have a life-length, L, with values in $[0, \infty)$ and probability distribution $G(x) = P(L \leqslant x)$. At the end of its life, it is transformed into N particles according as a probability law $\{p_n\colon n \geqslant 0\}$ on $\mathbb{N} = \{0, 1, \ldots\}$: $P(N = n) = p_n, \sum_{n \in \mathbb{N}} p_n = 1$. These new particles are taken to have the same life-length distribution and transformation probabilities as the original one. We

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assume that the life-length distribution and the transformation probabilities for each particle is independent of its time of birth and the number of other particles existing at the time. Let $Z(t)$ be the number of particles existing at time $t \geqslant 0$. As usual, we assume

$$p_0 + p_1 < 1, 1 < EN < \infty \text{ and } P(L = 0) < 1. \tag{0.1}$$

It is well known (see, for example, Athreya and Kaplan, 1976) that the limit

$$Z = \lim_{t \to \infty} Z(t)/EZ(t) \tag{0.2}$$

exists almost surely, and its Laplace transform $\phi(s) = E\mathrm{e}^{-sZ}$ satisfies the functional equation

$$\phi(s) = \int_0^\infty f(\phi(s\mathrm{e}^{-\alpha x}))\,\mathrm{d}G(x), \quad s > 0, \tag{0.3}$$

where $f(x) = \sum_{n=0}^\infty p_n x^n$ is the probability generating function of N, and α is the Malthus parameter defined by $(EN)\int_0^\infty \mathrm{e}^{-\alpha x}\,\mathrm{d}G(x) = 1$. It is known that $EZ = 1$ if $EN \log^+ N < \infty$ (as usual, we write $\log^+ x = \log x$ if $x \geqslant 1$, and $=0$ if $0 \leqslant x < 1$), and $Z=0$ almost surely otherwise; when $EN \log^+ N = \infty$, there are positive constants $\{C(t)\}$ such that

$$\lim_{t \to \infty} Z(t)/C(t) = W \quad \text{almost surely}$$

for some non-degenerate random variable W whose Laplace transform ϕ satisfies (0.3). [Cf. Cohn, 1982 or Schuh, 1982.] $\{Z(t)/C(t)\}$ is usually referred as the Seneta–Heyde norming of $Z(t)$.

We are interested to asymptotic properties of the random variable Z and W. In fact, we shall study the problems in a more general setting, namely for limits of martingales arising in homogeneous branching random walks.

A homogeneous branching random walk can be described as follows. Let $\mathbb{N}^* = \{1, 2, \ldots\}$ be the set of positive integers and let $\mathbb{U} = \{\emptyset\} \cup \bigcup_{k=1}^\infty (\mathbb{N}^*)^k$ be the set of all finite sequences including the null sequence $\emptyset$. Let $(\Omega, \mathbb{F}, P)$ be a probability space and let $\{N_u\colon u \in \mathbb{U}\}$ and $\{L_u\colon u \in \mathbb{U}\}$ be two independent families of independent random variables defined on $(\Omega, \mathbb{F}, P)$, the $N_u's$ are distributed as $N = N_\emptyset$ and with values on $\{0, 1, \ldots\}$, and the $L_u's$ are distributed as $L = L_\emptyset$ and with values on $\mathbb{R} = (-\infty, \infty)$. ($L$ is now not necessarily non-negative.) Let $\mathbb{T} = \mathbb{T}(\omega)$ be the Galton–Watson tree with defining elements $\{N_u\}$ – we have $\emptyset \in \mathbb{T}$ and, if $u \in \mathbb{T}$ and $i \in \mathbb{N}$, then $ui \in \mathbb{T}$ if and only if $1 \leqslant i \leqslant N_u$. The initial particle $\emptyset \in \mathbb{T}$ is placed at $S_\emptyset = 0$ of the real line $\mathbb{R} = (-\infty, \infty)$. It gives birth to $N_\emptyset$ new particles i $(1 \leqslant i \leqslant N_\emptyset)$ with displacements $L_\emptyset$. In general, if $u = u_1 \ldots u_n \in \mathbb{T}$ is a particle in nth generation ($u = \emptyset$ if $n = 0$), then the position of their children ui is given by

$$S_{ui} = L_\emptyset + L_{u_1} + \cdots + L_{u_1 \ldots u_n}, \quad 1 \leqslant i \leqslant N_u.$$

(Note that S_{ui} depends only on u, but not on i; by convention, $\emptyset i = i$.) Assume (0.1) and let $\alpha \in \mathbb{R}$ be such that $m(\alpha) := EN E\mathrm{e}^{-\alpha L} < \infty$. Then the sequence

$$Y_n := m(\alpha)^{-n} \sum_{u = u_1 \ldots u_n \in T} \mathrm{e}^{-\alpha S_u}, \quad n \geqslant 1,$$

is a martingale, so that the limit

$$Z := \lim_{n\to\infty} Y_n \tag{0.4}$$

exists almost surely. By considering the sub-trees beginning at $i \in \{1,\dots,N\}$, we see easily that Z satisfies the distributional equation

$$Z = A(Z_1 + \cdots + Z_N), \tag{0.5}$$

where $A = \mathrm{e}^{-\alpha L}/m(\alpha)$ – so that $ENEA = 1$ – $Z_1, Z_2, \dots$ are independent copies of Z and are also independent of (N, A). In terms of characteristic function or Laplace transform, the equation reads

$$\phi(t) = Ef(\phi(At)), \tag{0.5a}$$

where $\phi(t) = E\mathrm{e}^{\mathrm{i}tZ} (t \in \mathbb{R})$ or $E\mathrm{e}^{-tZ} (t > 0)$, f denotes always the probability generating function of N. Let Z be the random variable defined by (0.4). It is known – see, for example, Liu (1997a) – that the distribution of Z is the unique solution of (0.5) with mean 1 if

$$EN \log^+ N < \infty \quad \text{and} \quad EA \log^+ A < \infty \quad \text{with } EA \log A < 0 \tag{0.6}$$

and $Z = 0$ almost surely otherwise. We remark that Eq. (0.3) is a special case of (0.5a) with

$$P(L \geqslant 0) = 1, \quad \alpha > 0 \quad \text{and} \quad m(\alpha) = 1 \tag{0.7}$$

(so that $0 < A \leqslant 1$ almost surely). Therefore, if (0.7) holds, then the distribution of (0.4) is the same as that of (0.2). We assume (0.7) whenever (0.2) is concerned.

Assume (0.6) and let Z be the random variable defined by (0.2) or (0.4).

We first give a sufficient condition for the distribution of Z to have a density function with k-fold continuous derivatives on $(0, \infty)$ (Theorem 1.1). In particular, it turns out that the distribution of (0.4) has always a continuous density function; this is also a consequence of a theorem of Biggins and Grey (1979), and extends a result of Athreya (1969) saying that the distribution of (0.2) has always a continuous density function.

We next study the decay rate of the distribution function $P(Z \leqslant x)$ as $x \to 0$, and that of the characteristic function $\phi(t) = E\mathrm{e}^{\mathrm{i}tZ}$ or its derivative $\phi'(t)$ as $|t| \to \infty$. For example, we show that, as $x \to 0$ (i) if $p_0 = p_1 = 0$, then $P(Z \leqslant x)$ decays at a polynomial rate if and only if the same is true for $P(A \leqslant x)$ (Theorem 1.2); (ii) if $p_0 + p_1 > 0$, then $P(Z \leqslant x)$ decays always at a polynomial rate (Theorem 1.3 and Corollary 1.1); (iii) if $p_0 = p_1 = 0$ and if additionally $A \geqslant a$ for some constant $a > 0$, then $P(Z \leqslant x)$ decays at an exponential rate (Theorem 1.4). Note that in the context of Walton–Watson process – thus $L = 1$ almost surely – case (i) cannot occur. This difference between the Bellman–Harris process and the Galton–Watson process is remarkable. Athreya and Ney (1972, p. 178 open problem 13) asked about the relation between the distribution of $\lim Z(t)/EZ(t)$ and that of the embedded Galton–Watson process. Our results show that their decay rates at 0 are quite different if $p_0 = p_1 = 0$ and L has an exponential right tail, but similar if $p_0 + p_1 > 0$ or $L \leqslant M$ almost surely for some constant $M > 0$. For the special age-dependent process with $f(t) = t^2$, Bellman and Harris (1952) have studied the decay rate of the characteristic function $\phi(t) = E\mathrm{e}^{\mathrm{i}tZ}$

and that of its derivative $\phi'(t)$ as $|t| \to \infty$. Their results are extended and improved here. In particular, we prove that we have always $\phi'(t) = \mathrm{O}(|t|^{-1})$, which leads to an alternative proof for the result of Athreya (1969) about the absolute continuity on $(0,\infty)$ of the distribution of Z (Remark 1.1).

We then study the moments and polynomial right tail behavior of Z (Theorem 1.5). This study reveals an essential difference between the two cases $P(A \leqslant 1) = 1$ and $P(A \leqslant 1) < 1$. In the first case, the right tail probability $P(Z > x)$ $(x \to \infty)$ of Z behaves similarly as that of A: for all $p > 1$, $E[Z^p] < \infty$ if and only if $E[A^p] < \infty$; in the second case, there is usually a number $\chi > 1$ defined by $ENEA^{\chi} = 1$, such that $P(Z > x)$ behaves as $x^{-\chi}$ when $x \to \infty$. The conclusion in the first case was proved by Bingham and Doney (1975). But even in this special case, our new method remains interesting because of its simplicity; moreover, our method works in both cases $P(A \leqslant 1) = 1$ and $P(A \leqslant 1) < 1$, and it seems that the approach of Bingham and Doney (1975) cannot be extended to the case where $P(A \leqslant 1) < 1$ without additional conditions.

We finally study the exponential moments and the analyticity of the characteristic function ϕ of Z (Theorems 1.6), and the exponential decay rate and the growth order and type of the entire characteristic function (Theorem 1.7). After proving the analyticity of ϕ in the case where $P(A \leqslant 1) = 1$ and $P(N = 2) = 1$, Bellman and Harris (1952, p. 294) presumed that the analyticity would also holds if $P(A \leqslant 1) = 1$ and $E\mathrm{e}^{tN} < \infty$ for some $t > 0$. Here we give a necessary and sufficient condition for the analyticity in our extended case. In the case where $\operatorname{ess\,sup} N < \infty$ and $\operatorname{ess\,sup} A < 1$, we prove that Z has an entire characteristic function with order $\bar{\gamma} > 1$ (determined explicitly) and finite type, and that the right tail probability $P(Z > x)$ lies between $\exp(-C_i x^{\bar{\beta}})$ for some explicit $\bar{\beta} > 1$ and some constants $C_i \in (0,\infty)$, $i = 1, 2$.

Our theorems will be established for *any non-trivial solution* (*with or without mean*) of an associated functional equation extending (0.5a), and are therefore applicable for other limit variables arising in age-dependent branching processes and in homogeneous branching random walks. Let us explain this by an example: Cohn (1982) and Schuh (1982) proved that if the Bellman–Harris process $(Z(t))$ satisfies (0.1) with $EN \log^+ N = \infty$, then the limit variable W of the Seneta–Heyde norming has a continuous distribution on $(0,\infty)$; our study on the decay rate of characteristic function shows that the distribution of W is, in fact, *absolutely continuous* on $(0,\infty)$ under a simple moment condition on L (Remark 1.2).

For Galton–Watson processes, the problems were studied, for example, by Harris (1948), Dubuc (1971), Athreya (1972) and Bingham (1988).

In closing this section, we point out that the arguments of this paper can be applied to the study of general age-dependent branching processes, extended multiplicative cascades and general branching random walks: see Liu (1997b, 1999a,b).

1. Main results

In this section, unless otherwise specified, for simplicity we assume (0.1) and (0.6) and let Z be the random variable defined by (0.2) or (0.4). Then $q = P(Z = 0)$ is the unique fixed point in $[0,1)$ of f. We recall that $A = e^{-\alpha L}/m(\alpha)$, which reduces to $e^{-\alpha L}$ with $L \geqslant 0$ almost surely and $\alpha > 0$ if (0.2) is concerned. As usual, for two functions g and h, we write $g(t) = O(h(t))$ (resp. $o(h(t))$) if $\limsup |g(t)/h(t)| < \infty$ (resp. $=0$). If X is a random variable, we write P_X for its law and write $||X||_p$ for its L^p norm – $||X||_p = (E[|X|^p])^{1/p}$ if $p \geqslant 1$ and $||X||_p = E[|X|^p]$ if $0 < p < 1$ – and we define $||X||_\infty = \operatorname{ess\,sup} |X|$ and $||X||_{-\infty} = \operatorname{ess\,inf} |X|$. For $k \geqslant 0$, let C^k be the class of functions with k-fold continuous derivatives.

Theorem 1.1 (Existence of a density of class C^{k}). *Write $\phi(t) = Ee^{itZ}$, $t \in \mathbb{R}$ and let ϕ' be its derivative. For all $a \geqslant 0$, if either* (i) $p_0 = p_1 = 0$ *and* $E[A^{-a}] < \infty$, *or* (ii) $p_0 + p_1 > 0$ *and* $f'(q)E[A^{-a}] < 1$, *then the function $t \mapsto \phi'(t)t^a$ is integrable on $\mathbb{R}$, so that on* $(0,\infty)$, *the distribution of Z has a density function of class $C^{[a]}$ (where $[a]$ denotes the integral part of* (a)), *given by*

$$x \mapsto \frac{1}{2i\pi x} \int_{\mathbb{R}} e^{-itx} \phi'(t)\, dt, \quad x > 0. \tag{1.1}$$

Since condition (i) *or* (ii) *holds automatically for $a = 0$, we obtain immediately*:

Corollary 1.1 (Absolute continuity). *The distribution of Z is absolutely continuous on* $(0,\infty)$, *with a continuous density given by* (1.1).

In the context of age-dependent branching process (thus $P(A \leqslant 1) = 1$), this result was first claimed by Harris (1963) under the additional moment conditions that $EN^2 < \infty$ and $EA^{-\varepsilon} < \infty$ for some $\varepsilon > 0$; Athreya (1969) showed that these moment conditions can be removed. The result has been extended by Biggins and Grey (1979) to general branching random walks.

The following two theorems concern the decay rate of left tail distribution and characteristic function of Z. Recall that in the Galton–Watson process, there are two essentially different cases, according as $p_0 + p_1 = 0$ or not. Our results show that the same is true for the Bellman–Harris process. (We mention that there is a similar phenomenon for the limit variable of Mandelbrot's cascades: see Barral, 1997; Liu, 1997b.)

Theorem 1.2 (Left tail and decay rate of characteristic function: the case where $p_0 = p_1 = 0$). *Assume $p_0 = p_1 = 0$. Then*:

(i) *For all $a \in (0,\infty)$, $P(Z \leqslant x) = O(x^a)(x \to 0)$ if and only if $P(A \leqslant x) = O(x^a)$ $(x \to 0)$.*

(ii) *If for some constants $a > 0, C_i > 0$ $(i = 1,2)$ and all $x > 0$ small enough, $C_1 x^a \leqslant P(A \leqslant x) \leqslant C_2 x^a$, then the same is true for Z: there are some constants $K_i > 0$ $(i = 1,2)$ such that for all $x > 0$ small enough,*

$$K_1 x^a \leqslant P(Z \leqslant x) \leqslant K_2 x^a.$$

(iii) *For all $a \in [0,\infty)$, if $P(A \leqslant x) = O(x^a)(x \to 0)$, then as $|t| \to \infty$,*

$$|Ee^{itZ}| = O(|t|^{-a}) \quad \textit{and} \quad \frac{dEe^{itZ}}{dt} = O(|t|^{-(1+a)}).$$

Let us make some comments in the case of age-dependent processes. We remark that the condition $P(A \leqslant x) = O(x^a)(x \to 0)$ reduces to $P(L > y) = O(e^{-a\alpha y})(y \to \infty)$.

If $f(t)=t^2$, the first assertion of part (iii) was proved by Bellman and Harris (1952), while the second one improves their conclusion that if $P(A\leqslant x)=O(x^a)$ $(x\to 0)$ for some $a>0$, then $\mathrm{d}Ee^{\mathrm{i}tZ}/\mathrm{d}t=O(|t|^{-(1+\delta)})$ for some $\delta>0$. Our results show that, the small values of Z depend considerably on large values of L: for example $P(Z\leqslant x)=O(x^a)(x\to 0)$ if and only if $P(L>y)=O(e^{-a\alpha y})(y\to\infty)$. This is an interesting phenomena, remarking that the large values of Z depend only on large values of N: for example, for all $p>1$, we have $E[Z^p]<\infty$ if and only if $E[N^p]<\infty$: see Theorem 1.5(i) in the below.

Theorem 1.3 (Left tail and decay rate of characteristic function: the case where $p_0+p_1>0$). *Assume $p_0+p_1>0$ and let q be the unique fixed point in* $[0,1)$ *of* f. *Then*:

(i) *For all* $a\in(0,\infty)$, if $f'(q)E[A^{-a}]<1$, *then* $P(0<Z\leqslant x)=O(x^a)(x\to 0)$; *conversely, if* $E[Z^{-a}\,|Z>0]<\infty$, *then* $f'(q)E[A^{-a}]<1$. *In the case where* (0.2) *is concerned or in the case where* $A\leqslant 1$ *almost surely, the assertion can be improved to the following*: *for all* (*fixed*) $a\in(0,\infty)$, $E[Z^{-a}\,|Z>0]<\infty$ *if and only if* $f'(q)E[A^{-a}]<1$.

(ii) *For all* $a\in[0,\infty)$, *if* $f'(q)E[A^{-a}]<1$, *then as* $|t|\to\infty$,

$$|E(e^{\mathrm{i}tZ}\,|Z>0)|=O(|t|^{-a})\quad\textit{and}\quad\left|\frac{\mathrm{d}E(e^{\mathrm{i}tZ}\,|\,Z>0)}{\mathrm{d}t}\right|=O(|t|^{-(a+1)}).$$

The following result is immediate by Theorem 1.3(i).

Corollary 1.2. *Assume* $p_0+p_1>0$ *and that for some* $\beta>0$,

$$f'(q)E[A^{-\beta}]=1.$$

Then for all $0<a<\infty$, $E[Z^{-a}\,|\,Z>0]<\infty$ *if and only if* $a<\beta$.

Remark 1.1. Of course, the condition in part (iii) of Theorem 1.2 and that in part (ii) of Theorem 1.3 are automatically satisfied for $a=0$, so that in both cases $p_0=p_1=0$ and $p_0+p_1>0$, we have

$$\left|\frac{\mathrm{d}E(e^{\mathrm{i}tZ})}{\mathrm{d}t}\right|=O(|t|^{-1})\quad(|t|\to\infty).$$

In particular, the function $|\mathrm{d}E(e^{\mathrm{i}tZ})/\mathrm{d}t|$ is square integrable on $\mathbb{R}$. This gives *an alternative proof for the absolute continuity on* $(0,\infty)$ *of the distribution of* Z, remarking that the function $t\mapsto \mathrm{d}E(e^{\mathrm{i}tZ})/\mathrm{i}\mathrm{d}t$ is the characteristic function of the probability measure $xP_Z(\mathrm{d}x)$ (P_Z denoting the distribution of Z).

Remark 1.2. Except for the results on the derivative of characteristic function, all the conclusions in Theorems 1.2 and 1.3 and Corollary 1.2 also hold for *any non-trivial solution* Z (*with or without mean*) of (0.5) without assuming (0.6). In particular, if for some $a>\frac{1}{2}$, either (a) $p_0=p_1=0$ and $EA^{-a}<\infty$, or (b) $p_0+p_1>0$ and $f'(q)EA^{-a}<1$, then $|E(e^{\mathrm{i}tZ}\,|Z>0)|$ is square integrable on $\mathbb{R}$, so that the distribution of Z is absolutely continuous on $(0,\infty)$.

Theorem 1.4 (Left tail and decay rate of characteristic function: the exponential case). *Assume that either N or A is not almost surely constant, and that* $||N||_{-\infty} := \operatorname{ess\,inf} N > 1$ *and* $||A||_{-\infty} := \operatorname{ess\,inf} A > 0$. *Write* $\gamma = -\log ||N||_{-\infty} / \log ||A||_{-\infty}$. *Then* $0 < \gamma < 1$ *and the following assertions hold*:

(i) *For all* $\varepsilon > 0$, *there are some constants* $C_1 > 0$ *and* $C_2 > 0$ *such that for all* $x > 0$ *small enough*,

$$\exp\{-C_1 x^{-(\gamma/(1-\gamma)+\varepsilon)}\} \leqslant P(Z \leqslant x) \leqslant \exp\{-C_2 x^{-(\gamma/(1-\gamma))}\};$$

moreover, the conclusion is also valid for $\varepsilon = 0$ *if additionally* $P(A = ||A||_{-\infty}) > 0$.

(ii) *For some constant* $K > 0$ *and all* $t \in \mathbb{R}$ *with* $|t| \geqslant 1$,

$$|E\mathrm{e}^{\mathrm{i}tZ}| \leqslant \mathrm{e}^{-K|t|^{\gamma}}.$$

If $L = 1$ almost surely, part (i) is due to Harris (1948), and part (ii) can be deduced – see Bingham (1988) – from a result, also due to Harris (1948), concerning the decay rate of $E\mathrm{e}^{-tZ}$ $(t \to \infty)$.

In the context of age-dependent processes, Theorem 1.4 shows that if $||N||_{-\infty} := \operatorname{ess\,inf} N$ $(=m) \geqslant 2$ and $||L||_{\infty} = \operatorname{ess\,sup} L < \infty$, then the left tail $P(Z \leqslant x)$ has an exponential decay rate with exponent $\beta := \gamma/(1-\gamma)$, where $\gamma = (\log ||N||_{-\infty})/(\alpha ||L||_{\infty})$. It is interesting to remark that a similar result holds for the right tail behavior of Z: in Theorem 1.7 below, we shall see that if $||N||_{\infty} = \operatorname{ess\,sup} N < \infty$ and $||L||_{-\infty} = \operatorname{ess\,inf} L > 0$, then as $x \to \infty$, the right tail probability $P(Z > x)$ has an exponential decay rate with exponent $\bar{\beta} = \bar{\gamma}/(\bar{\gamma} - 1)$, where $\bar{\gamma} = \log ||N||_{\infty})/(\alpha ||L||_{-\infty})$.

Until now, we have studied the similar properties of (0.2) and (0.4). The following theorem shows that a new phenomena arises for (0.4) in the case where $P(A > 1) > 0$. In this case, the equation $ENE[A^x] = 1$ has usually two solutions in $[1, \infty)$: the solution 1 and another solution $\chi > 1$. The problem will be called *lattice* if for some $h > 0$, $\log A$ is almost surely an integer multiple of h whenever $A > 0$; the largest such h will be called the span. Otherwise, it is called *non-lattice*.

Theorem 1.5 (Moments and polynomial right tail). (i) *For each fixed* $p > 1$, $E[Z^p] < \infty$ *if and only if* $E[N^p] < \infty$ *and* $ENE[A^p] < 1$.

(ii) *Suppose that for some* $\chi > 1$,

$$ENE[A^{\chi}] = 1, \quad E[A^{\chi} \log^+ A] < \infty \quad \textit{and} \quad E[N^{\chi}] < \infty.$$

If the problem is non-lattice, then there is a constant $c \in (0, \infty)$ *such that*

$$\lim_{x \to \infty} x^{\chi} P(Z > x) = c;$$

if the problem is lattice, then

$$0 < \liminf_{x \to +\infty} x^{\chi} P(Z > x) \leqslant \limsup_{x \to +\infty} x^{\chi} P(Z > x) < \infty.$$

If $P(A \leqslant 1) = 1$, only case (i) occurs, and the assertion is, in fact, a consequence of a result of Bingham and Doney (1975). Even in this case, our new proof remains interesting, remarking that the proof of Bingham and Doney (1975) is not easy. (For the case where $P(A \leqslant 1) < 1$, the author required extra conditions on A to extend their approach.) Our results show that, if $P(A \leqslant 1) = 1$, then the right tail behavior of Z is similar to that of N; if $P(A \leqslant 1) < 1$, then the right tail of Z has a polynomial decay rate.

Theorem 1.6 (Exponential moments and analyticity of characteristic function). *The following assertions are equivalent*: (i) $A \leqslant 1$ *almost surely and for some* $t > 0$, $E\mathrm{e}^{NA} < \infty$; (ii) *for some* $t > 0, E\mathrm{e}^{tZ} < \infty$. *Consequently*, Z *has an analytic characteristic function if and only if so does* N, *together with* $P(A \leqslant 1) = 1$.

The analyticity of $E\mathrm{e}^{\mathrm{i}tZ}$ was obtained by Bellman and Harris (1952) in the case where $N = 2$ and $A \leqslant 1$ almost surely.

We recall that if f is an entire function, then its growth *order* is defined by

$$\rho = \limsup_{r\to\infty} \log\log M(r)/\log r, \quad \text{where } M(r) = \max_{|z|=r} |f(z)|;$$

if $\rho < \infty$, more detailed information on the growth rate is contained in the *type* τ (with respect to the order ρ) defined by

$$\tau = \limsup_{r\to\infty} \log M(r)/r^{\rho};$$

if we replace lim sup with lim inf, we obtain respectively the *lower order* $\underline{\rho}$ and the *lower type* $\underline{\tau}$ (with respect to the order ρ). The entire function f is called to be of regular growth if $\rho = \underline{\rho}$.

Theorem 1.7 (Exponential right tail and growth order of entire characteristic function). *Assume that either* N *or* A *is not almost surely constant, and that* $\|N\|_\infty := \operatorname{ess\,sup} N < \infty$ *and* $\|A\|_\infty := \operatorname{ess\,sup} A < 1$. *Write* $\bar{\gamma} = -\log\|N\|_\infty/\log\|A\|_\infty$. *Then* $\bar{\gamma} > 1$ *and the following assertions hold*:

(i) *For all* $\varepsilon > 0$, *there exist some constants* $C_i > 0$ $(i=1,2)$ *such that for all* $x > 0$ *small enough*,

$$\exp\{-C_1 x^{(\bar{\gamma}/(\bar{\gamma}-1)+\varepsilon)}\} \leqslant P(Z > x) \leqslant \exp\{-C_2 x^{(\bar{\gamma}/(\bar{\gamma}-1))}\}.$$

The assertion also holds for $\varepsilon = 0$ *if additionally* $P(A = \|A\|_\infty) > 0$.

(ii) Z *has an entire characteristic function of regular growth with order* $\bar{\gamma}$ *and finite type*; *the lower type is non-zero if additionally* $P(A = \|A\|_\infty) > 0$.

By Theorem 2.1 of Liu (1996b), the condition $P(A = \|A\|_\infty) > 0$ in the above theorem can be relaxed to the following: for some constants $c > 0$, $a \geqslant 0$ and all $x > 0$ sufficiently small,

$$P(A/\|A\|_\infty > 1 - x) \geqslant cx^a.$$

However, the new approach in this paper remains interesting since the proof in Liu (1996) is not easy.

It is interesting to compare Theorem 1.7(i) with Theorem 1.4(i): they are quite similar.

The results will be established in a more general setting. Theorem 1.1 follows from Theorem 3.2; Theorems 1.2 and 1.3 are consequences of Theorems 4.1, 5.1, Corollary 5.1 and Proposition 2.1; Theorem 1.4 comes from Corollary 6.1(i) and Theorem 6.1(iii), Theorem 1.5 is contained in Theorems 7.1 and 7.2; Theorem 1.6 follows from Theorem 8.1 and Corollary 8.1; Theorem 1.7 is in Corollary 8.2(i), (iv) and (v).

2. The associated functional equation

Our study will be based on the functional equation

$$\phi(s) = E[f(\phi(As)], \tag{2.1}$$

where $A \geqslant 0$ is an arbitrary non-negative random variable, f is a probability generating function, and ϕ is the Laplace transform or the characteristic function of a probability distribution on $\mathbb{R}_+ = [0, \infty)$. In all this paper, N stands for a random variable with probability generating function f. We remark that Eq. (0.3) is nothing but (2.1) with $A = e^{-\alpha L}$, $L \geqslant 0$, $\alpha > 0$, thus $0 < A \leqslant 1$ and $ENEA = 1$. For a number of applications in branching random walks, it is also interesting to study this equation without assuming $P(0 < A \leqslant 1) = 1$, neither $EAEN = 1$ nor $EN \log^+ N < \infty$. In terms of a distributional identity, Eq. (2.1) reads

$$Z \stackrel{d}{=} A(Z_1 + \cdots + Z_N), \tag{2.1a}$$

where $\stackrel{d}{=}$ means the equality in distribution, $N, A, Z_1, Z_2, \ldots$ are independent random variables, Z_i having the same distribution as Z. We say that a non-negative random variable Z is a solution of (2.1) if so is its Laplace transform or characteristic function ϕ.

Write $\alpha_0 = P(A = 0)$ and recall the notations $p_n = P(N = n)$, $n \in \mathbb{N}$. If Δ is a set or statement, we write $1\{\Delta\}$ for its indicator function. It is easily verified that the probability generating function of $\tilde{N} := N1\{A > 0\}$ is

$$\tilde{f}(t) = \alpha_0 + (1 - \alpha_0) f(t), \quad t \geqslant 0,$$

and the extinction probability $q = P(Z = 0)$ is the unique fixed point in $[0, 1)$ of $\tilde{f}$ provided that Z is a non-trivial solution of (2.1).

We first give a *principle of reduction*, showing that the case where $\alpha_0 + p_0 > 0$ can be reduced to the case where $\alpha_0 = p_0 = 0$.

Proposition 2.1. *Let $Z \geqslant 0$ be a random variable whose Laplace transform $\phi(t) = Ee^{-tZ}$ ($t \in \mathbb{R}_+$) is a non-trivial solution of* (2.1). *Let $\bar{A}$ be a random variable whose distribution is that of A conditional on $A > 0$, and for $t \in \mathbb{R}_+$, write*

$$\bar{\phi}(t) = E[e^{-tZ} | Z > 0] \quad \textit{and} \quad \bar{f}(t) = \frac{\tilde{f}((1-q)t + q) - q}{1 - q},$$

where $\tilde{f}$ and $q = P(Z = 0)$ are defined as above. Then Eq. (2.1) *reduces to*

$$\bar{\phi}(t) = E\bar{f}(\bar{\phi}(\bar{A}t)),$$

with $P(\bar{A}=0)=\bar{f}(0)=0$ *and* $\bar{f}'(0)=(1-\alpha)f'(q)$. *Moreover,* $\bar{f}'(0)=0$ *if and only if* $p_0=p_1=\alpha_0=0$.

The proof is easy, and is therefore omitted. From now on, for convenience we always assume

$$\alpha_0=0,\quad P(A=1)<1,\quad p_0+p_1<1 \quad \text{and} \quad EN<\infty, \tag{2.2}$$

unless otherwise specified. The following result concerns the *existence and uniqueness of solution* of (2.1).

Proposition 2.2. (i) *If*

$$EN>1 \quad and \quad ENEA^{\lambda}\leqslant 1 \quad for\ some\ \lambda\in(0,1], \tag{2.3}$$

then Eq. (2.1) *has a non-trivial solution* (*in the class of Laplace transforms of probability measures on* $\mathbb{R}_+$); *the converse holds if additionally*

$$EA\log^+ A<\infty. \tag{2.4}$$

(ii) *Eq.* (2.1) *has a non-trivial solution with finite mean if and only if*

$$ENEA=1,\ EN\log^+ N<\infty \ and\ EA\log^+ A<\infty \ with\ EA\log A<0; \tag{2.5}$$

when (2.5) *holds, all solutions have finite mean, and are parametrized by their means.* (*In particular, there is at most one solution with mean* 1.)

Part (i) follows from Theorem 1.1 of Liu (1998). In part (ii), the second assertion is a consequence of the first assertion and the uniqueness theorem of Biggins and Kyprianou (1997, Theorem 1.5), while the first comes from Theorem 1.1 of Liu (1997a).

We have already seen that if (2.5) holds, then the random variable Z defined by (0.4) is a solution of (2.1) with mean 1. In the case where (2.5) fails, (2.1) has no non-trivial solution with finite mean, but it does have non-trivial solution by Proposition 2.2. In fact, if

$$ENEA=1,\quad EN\log^+ N=\infty \quad \text{and} \quad EA\log^+ A<\infty \quad \text{with } EA\log A<0, \tag{2.6}$$

then there is a sequence of constants $\{c_n\}$ such that Y_n/c_n converges in probability to a solution Z' of (E) with $EZ'=\infty$ – see Biggins and Kyprianou (1997); if

$$ENEA=1,\quad E(N^{1+\delta})+E(A^{1+\delta})<\infty \quad \text{for some } \delta>0 \text{ and } EA\log A=0, \tag{2.7}$$

then

$$Y_n^* := \sum_{u=u_1\ldots u_n\in T(\omega)} X_u\log\frac{1}{X_u},\quad \text{where } X_u=m(\alpha)^{-n}\mathrm{e}^{-\alpha S_u}$$

is a martingale, and converges to a non-trivial solution Z^* of (2.1) with $\lim_{t\to 0}(1-\phi^*(t))/|t\log t|=1$, where $\phi^*(t)=E\mathrm{e}^{-tZ^*}$ – see Liu (1997b) or an independent work of Kyprianou (1999) on the derivative of the additive martingale in the branching random walk.

Many results proved in the following sections, such as Theorems 3.1, 4.1, 5.1, 6.1 and their corollaries, are all applicable for each of the random variables Z, Z' and Z^* mentioned above.

3. Absolute continuity [1]

We first prove that $\lim_{|t|\to\infty} E[e^{itZ}|Z>0]=0$ for any non-trivial solution of (2.1). The result was proved by Athreya (1969) in the case where $EN\log^+ N<\infty$ and $0<A\leqslant 1$ almost surely. The extension here will be applied later.

Theorem 3.1. *Assume* $p_0 = 0$ *and* (2.3), *and let* Z *be a non-trivial solution of* (2.1). *Then*

$$\lim_{|t|\to\infty} E(e^{itZ})=0.$$

Proof. We write $\phi(t)=E(e^{itZ})$ for $t\in\mathbb{R}$.

(i) We first prove that $\limsup_{|t|\to\infty}|\phi(t)|=0$ or 1. By (2.1),

$$|\phi(t)|\leqslant Ef(|\phi(At)|),\quad t\in\mathbb{R}. \tag{3.1}$$

Letting $|t|\to\infty$ and using Fatou's lemma, we obtain $l\leqslant f(l)$, where $l:=\limsup_{|t|\to\infty}|\phi(t)|$. Therefore $l=0$ or 1, noting that $f(x)<x$ if $0<x<1$.

(ii) We next prove that for all $t\neq 0$, $|\phi(t)|<1$. Otherwise, by Lemma 4 of Chapter IV.1 of Feller, there is some $h>0$ such that $|\phi(h)|=1$ and $|\phi(t)|<1$ if $0<t<h$. So

$$1=|\phi(h)|=|Ef(\phi(Ah))|\leqslant E|\phi(Ah)|.$$

Therefore, a.s. $|\phi(Ah)|=1$. Since $P(0<A<1)>0$, it follows that for some $0<a<1$, $|\phi(ah)|=1$, which is a contradiction with the definition of h.

(iii) We then show that $\limsup_{|t|\to\infty}|\phi(t)|<1$. Together with part (i), this will end the proof of theorem. Assume $\limsup_{t\to\infty}|\phi(t)|=1$. Let $0<t_0<\infty$ be arbitrary fixed, and let $0<\varepsilon<1-|\phi(t_0)|$. Choose $t_1=t_1(\varepsilon)$ and $t_2=t_2(\varepsilon)$ such that $0<t_1<t_0<t_2<\infty$,

$$|\phi(t_1)|=|\phi(t_2)|=1-\varepsilon\quad\text{and}\quad|\phi(t)|\leqslant 1-\varepsilon\quad\text{if } t\in[t_1,t_2]. \tag{3.2}$$

This is possible since ϕ is continuous and $|\phi(0)|=\limsup_{t\to\infty}|\phi(t)|=1$. By Eq. (2.1), for all $t\in\mathbb{R}$, $|\phi(t)|\leqslant E|\phi(At)|$. Let $A_i\,(i\geqslant 1)$ be independent copies of A, then by iteration,

$$|\phi(t)|\leqslant E|\phi(A_1\ldots A_n t)|,\quad n\geqslant 1,\ t\in\mathbb{R}.$$

Therefore by Jensen's inequality, for all $n\geqslant 1$ and all $t\in\mathbb{R}$,

$$f(|\phi(t)|)\leqslant f(E|\phi(A_1\ldots A_n t)|)\leqslant Ef(|\phi(A_1\ldots A_n t)|),$$

so that

$$Ef(|\phi(At)|)\leqslant Ef(|\phi(A_1\ldots A_{n+1}t)|),\quad n\geqslant 0,\ t\in\mathbb{R}.$$

[1] After the results had been achieved, the referee kindly attracted the author's attention to the work of Biggins and Grey (1979) about the continuity (and absolute continuity) of limit variables (with finite mean) in general branching random walks, where there are some ideas similar in nature; their main result implies that under (0.6), the distribution of Z defined by (0.4) has a continuous density function on $(0,\infty)$.

It follows (using (3.1)) that

$$\begin{aligned}1-\varepsilon &= |\phi(t_2)| \leqslant Ef(|\phi(At_2)|) \leqslant Ef(|\phi(A_1\dots A_n t_2)|)\\ &\leqslant f(1-\varepsilon)P[t_1 < A_1\dots A_n t_2 \leqslant t_2] + 1 - P[t_1 < A_1\dots A_n t_2 \leqslant t_2]\\ &= -[1-f(1-\varepsilon)]P[t_1 < A_1\dots A_n t_2 \leqslant t_2] + 1.\end{aligned}$$

Therefore,

$$\frac{1-f(1-\varepsilon)}{1-(1-\varepsilon)}P[t_1 < A_1\dots A_n t_2 \leqslant t_2] \leqslant 1.$$

Since $\lim_{\varepsilon\to 0} t_1(\varepsilon)=0$ (this can be easily verified) and $t_1(\varepsilon)/t_2(\varepsilon) \leqslant t_1(\varepsilon)/t_0 \to 0$ $(\varepsilon \to 0)$, letting $\varepsilon \to 0$ in the above inequality gives

$$f'(1)P[A_1\dots A_n \leqslant 1] \leqslant 1,$$

where $f'(1)$ is the left derivative of f at 1. Let $\lambda \in (0,1]$ be defined as in (2.3), so that $EA^\lambda \leqslant 1/EN < 1$. Then by Markov's inequality,

$$P[A_1\dots A_n > 1] = P[(A_1\dots A_n)^\lambda > 1] \leqslant [E(A^\lambda)]^n \to 0 \quad \text{as } n\to\infty.$$

So $\lim_{n\to\infty} P[A_1\dots A_n \leqslant 1] = 1$. Therefore, letting $n\to\infty$ in the preceding inequality on $f'(1)$ gives $f'(1) \leqslant 1$. This contradicts the hypothesis that $f'(1) = EN > 1$, and proves that $\limsup_{t\to\infty} |\phi(t)| < 1$. The same argument above applies for $\phi(-t)$ instead of $\phi(t)$, yielding that $\limsup_{t\to\infty}|\phi(-t)| < 1$. Therefore $\limsup_{|t|\to\infty}|\phi(t)| < 1$, and the proof is finished. □

Remark 3.1. If $\alpha_0 + p_0 > 0$, then by Theorem 3.1 and the principal of reduction (Proposition 2.1), we have

$$\lim_{|t|\to\infty} Ee^{itx} = q,$$

$q = P(Z=0)$ being the unique fixed point in $[0,1)$ of $\tilde{f}$.

We now prove that any solution with finite mean is absolutely continuous on $(0,\infty)$, and give a sufficient condition for the density to have *k-fold* continuous derivatives, $k\in\mathbb{N}$. The absolute continuity was proved by Athreya (1969) in the case where $A \leqslant 1$ almost surely. The argument is inspired by that of Athreya (1969).

Theorem 3.2. *Assume* (2.5) *and let* Z *be a non-trivial solution of* (2.1). *Write* $\phi(t) = Ee^{itZ}$, $t\in\mathbb{R}$. *Then* ϕ' *is integrable on* $\mathbb{R}$, *and the distribution of* Z *has a continuous density function on* $(0,\infty)$, *given by*

$$x \mapsto \frac{1}{2i\pi x}\int_{\mathbb{R}} e^{-itx}\phi'(t)\,dt, \quad x>0. \tag{3.3}$$

Moreover, for all $p\geqslant 0$, *if* $p_1 EA^{-p} < 1$ (*it is interpreted as* $EA^{-p} < \infty$ *if* $p_1 = 0$), *then the function* $\phi_p(t) = \phi'(t)t^p$ *is integrable on* $\mathbb{R}$, *so that on* $(0,\infty)$, *the distribution of* Z *has an density of class* $C^{[p]}$ *given by* (3.3).

Proof. We remark that hypothesis (2.5) ensures $EZ < \infty$. For simplicity, we assume $EZ = 1$. Therefore

$$\phi'(t)/i = \int_{\mathbb{R}} e^{itx} x\, P_Z(dx)$$

is the characteristic function of the probability measure $xP_Z(\mathrm{d}x)$. So by the inversion formula for Fourier transforms, we need only prove that for all $p \geqslant 0$, if $p_1 EA^{-p} < 1$, then the function ϕ_p is integrable on $\mathbb{R}$, remarking that the preceding condition $p_1 EA^{-p} < 1$ is automatically satisfied for $p=0$. Let $\theta \in (p_1, 1)$ be such that $\theta EA^{-p} < 1$ and let $\tau > 0$ be sufficiently large such that

$$f'(|\phi(t)|) \leqslant \theta \quad \text{if } |t| \geqslant \tau.$$

This is possible since $f'(0) = p_1 < 1$ and $\lim_{|t|\to\infty} |\phi(t)| = 0$ by Theorem 3.1. By Eq. (2.1), for all $t \in \mathbb{R}$,

$$\phi'(t) = E[f'(\phi(At))\phi'(At)A].$$

So

$$|\phi'(t)| \leqslant E[f'(|\phi(At)|)|\phi'(At)|A], \quad t \in \mathbb{R}.$$

For all $T > 0$, put

$$M_T = \int_\tau^T |\phi'(t)| t^p \, \mathrm{d}t \quad \text{if } \tau \leqslant T, \quad \text{and} \quad M_T = 0 \quad \text{if } \tau > T.$$

Now let $T > \tau$ be arbitrarily fixed. Then if $\tau \leqslant T$,

$$\begin{aligned} M_T &\leqslant E \int_\tau^T f'(|\phi(At)|)|\phi'(At)|A t^p \, \mathrm{d}t \\ &= E\left[A^{-p} \int_{\tau A}^{TA} f'(|\phi(x)|) x^p \, \mathrm{d}x\right]. \end{aligned}$$

Since $|\phi'(x)| \leqslant EZ = 1$ and $\int_{\tau A}^{TA} \leqslant 1\{\tau A \leqslant \tau\} \int_{\tau A}^{\tau} + 1\{\tau \leqslant TA\} \int_\tau^{TA}$, it follows that

$$M_T \leqslant C + \theta E\left[1\{\tau \leqslant TA\} A^{-p} \int_\tau^{TA} |\phi'(x)| x^p \, \mathrm{d}x\right],$$

where $C = E[1\{A \leqslant 1\}A^{-p}] f'(1) \tau^{p+1}/(p+1) < \infty$ is independent of T. The inequality holds evidently if $\tau > T > 0$. Therefore, for all $T > 0$,

$$M_T \leqslant C + \theta E \frac{M_{TA}}{A^p}.$$

Let $\{A_i\colon i \geqslant 1\}$ be independent copies of A. By iteration, for all $n \geqslant 1$,

$$M_T \leqslant C(1 + \theta E[A^{-p}] + \cdots + (\theta E[A^{-p}])^{n-1}) + \theta^n E \frac{M_{TA_1\ldots A_n}}{A_1^p \ldots A_n^p}.$$

Because

$$\begin{aligned} \theta^n E \frac{M_{TA_1\ldots A_n}}{A_1^p \ldots A_n^p} &= \theta^n E\left[1\{\tau \leqslant TA_1 \ldots A_n\} \int_\tau^{TA_1\ldots A_n} t^p \, \mathrm{d}t/(A_1^p \ldots A_n^p)\right. \\ &\leqslant \theta^n T^{p+1} (EA)^n/(p+1) \;\to 0 \quad \text{as } n \to \infty \end{aligned}$$

(recall that $EA = 1/EN < 1$ by (2.5)), letting $n \to \infty$ in the preceding inequality on M_T gives

$$M_T \leqslant C/(1 - \theta E[A^{-p}]).$$

The right side being independent of T, by letting $T \to \infty$ and using the monotone convergence theorem we obtain

$$\int_{\tau}^{\infty} |\phi'(t)|\, t^p \,\mathrm{d}t \leqslant C/(1-\theta E[A^{-p}]) < \infty.$$

The same argument applies for $\phi'(-t)$ instead of $\phi'(t)$, showing that $\int_{\tau}^{\infty} |\phi'(-t)|\, t^p \,\mathrm{d}t < \infty$. So $\int_{|t|\geqslant\tau} |\phi'(t)| \,\mathrm{d}t < \infty$, and the proof is finished. □

4. Left tail and decay rate of $E\mathrm{e}^{\mathrm{i}tZ}$ and its derivative: the case where $p_0 = p_1 = 0$

We first establish some lemmas.

Lemma 4.1. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded function and let A be a positive random variable such that for some constants $p \in (0,1)$, $a \in (0,\infty)$, C, $t_0 \in [0,\infty)$ and all $t > t_0$*

$$\phi(t) \leqslant pE\phi(At) + Ct^{-a}.$$

If $pE(A^{-a}) < 1$, then as $t \to \infty$, $\phi(t) = \mathrm{O}(t^{-a})$.

Proof. We can assume $t_0 = 0$ by taking C large enough if necessary. Let $\{A_i\}$ be independent copies of A. Then for all $t > 0$ $\phi(t) \leqslant pE\phi(A_1 t) + Ct^{-a}$. By induction, for all $n \geqslant 1$ and all $t > 0$,

$$\phi(t) \leqslant p^n E\phi(A_1 \ldots A_n t) + Ct^{-a}[1 + pE(A^{-a}) + \cdots + (pE(A^{-a}))^{n-1}]. \tag{4.1}$$

In fact if (4.1) holds for some $n \geqslant 1$, then it also holds for $n+1$ because

$$\phi(A_1 \ldots A_n t) \leqslant p \int \phi(A_1 \ldots A_n x t) P_A(\mathrm{d}x) + C(A_1 \ldots A_n t)^{-a}$$

(with P_A denoting the law of A), and so

$$E\phi(A_1 \ldots A_n t) \leqslant pE\phi(A_1 \ldots A_n A_{n+1} t) + Ct^{-a}(E(A^{-a}))^n.$$

Therefore (4.1) holds for all $n \geqslant 1$. Letting $n \to \infty$ we see that for all $t > 0$, $\phi(t) \leqslant Ct^{-a}/[1 - E(A^{-a})]$. □

Lemma 4.2. *Let $Z \geqslant 0$ be a non-trivial solution of* (2.1). *Assume* $p_0 = 0$, *and write* $m = \min\{i \geqslant 1\colon\ p_i > 0\}$. *Then the following assertions hold*:

(i) *For all* $a > 0$, *if* $EZ^{-a} < \infty$, *then* $EA^{-a} < \infty$; *if additionally* $0 < p_1 < 1$, *then* $p_1 EZ^{-a} < 1$.

(ii) *For all* $K > 0$ *and all* $x > 0$, $P(Z \leqslant x) \geqslant p_m [P(Z \leqslant K)]^m P[A \leqslant x/(Km)]$.

Proof. By Eq. (2.1a), we obtain

$$EZ^{-a} = \sum_{n=m}^{\infty} EA^{-a} E[(Z_1 + \cdots + Z_n)^{-a}] p_n \geqslant EA^{-a} E[(Z_1 + \cdots + Z_m)^{-a}] p_m.$$

This gives $EA^{-a} < \infty$. In the case where $0 < p_1 < 1$, it is easily seen that $EZ^{-a} > EA^{-a}E[(Z_1)^{-a}]p_1$, so $p_1 EZ^{-a} < 1$. This ends the proof of part (i). The proof of part (ii) is similar, remarking that

$$\begin{aligned} P(Z \leqslant x) &= P[A(Z_1 + \cdots + Z_N) \leqslant x] \\ &\geqslant P[N = m, A \leqslant x/(Km) \text{ and } Z_i \leqslant K \text{ for all } 1 \leqslant i \leqslant m]. \qquad \square \end{aligned}$$

Lemma 4.3. *If* $P(A \leqslant x) = \mathrm{O}(x^a)$ $(x \to 0)$ *for some* $a > 0$, *then for all* $b > a$,

$$E[(1 + At)^{-b}] = \mathrm{O}(t^{-a}) \quad \text{as } t \to \infty.$$

Proof. Let $C > 0$ be a constant such that for all $x \geqslant 0$, $P(A \leqslant x) \leqslant Cx^a$. Then for all $t > 0$,

$$\begin{aligned} E[(1 + At)^{-b}] &= \int_0^1 P[(1 + At)^{-b} \geqslant x]\,\mathrm{d}x = \int_0^1 P[A \leqslant (x^{-1/b} - 1)/t]\,\mathrm{d}x \\ &\leqslant Ct^{-a} \int_0^1 (x^{-1/b} - 1)^a\,\mathrm{d}x = Ct^{-a} b \int_0^1 \frac{(1-y)^a}{y^{1+a-b}}\,\mathrm{d}y, \end{aligned}$$

where the last equality holds by change of variables $x = y^b$. Since the last integral is finite when $0 < a < b$, the proof is finished. $\square$

The following Tauberian result will be frequently used; for a slightly weaker version, see Barral (1997, Lemme 1.4.1.d).

Lemma 4.4. *Let* $X > 0$ *be a positive random variable. For all* $0 < a < \infty$, *consider the following statements*:

(i) $E[X^{-a}] < \infty$; (ii) $E[\mathrm{e}^{-tX}] = \mathrm{O}(t^{-a})$, $t \to \infty$;
(iii) $P[X \leqslant x] = \mathrm{O}(x^a)$, $x \to 0$; (iv) $\forall b \in (0, a)$, $E[X^{-b}] < \infty$.

Then the following implications hold: (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv).

Proof. The implications "(ii) $\Rightarrow$ (iii)" and "(i) $\Rightarrow$ (iii)" are easy: by Markov's inequality, we have for all $0 < x < \infty$,

$$P[X \leqslant x] = P[\mathrm{e}^{-X/x} \geqslant \mathrm{e}^{-1}] \leqslant \mathrm{e}E[\mathrm{e}^{-X/x}]$$

and

$$P[X \leqslant x] = P[X^{-a} \geqslant x^{-a}] \leqslant x^a E[X^{-a}].$$

Assume (iii) and let $C > 0$ be such that for all $x \leqslant 1$, $P[X \leqslant x] \leqslant Cx^a$. Then for all $0 < b < a$,

$$E[X^{-b}] = b \int_0^\infty y^{b-1} P[X^{-1} > y]\,\mathrm{d}y \leqslant b \int_0^1 y^{b-1}\,\mathrm{d}y + b \int_1^\infty C y^{b-1-a}\,\mathrm{d}y < \infty.$$

We have therefore proved that (iii) $\Rightarrow$ (iv). It remains to prove that (iii) $\Rightarrow$ (ii). Let $C > 0$ be as above and write $t = 1/x$ for $x \in (0, \infty)$. Then

$$E[\mathrm{e}^{-tX}] = \int_{[0,x]} \mathrm{e}^{-ty} P_X(\mathrm{d}y) + \sum_{n=0}^{\infty} \int_{2^n x}^{2^{n+1} x} \mathrm{e}^{-ty} P_X(\mathrm{d}y)$$

$$\leqslant P_X[0,x] + \sum_{n=0}^{\infty} \mathrm{e}^{-t2^n x} P_X[0, 2^{n+1}x]$$

$$\leqslant Cx^a + C\sum_{n=0}^{\infty} \mathrm{e}^{-2^n}(2^{n+1})^a x^a = C\left(1 + \sum_{n=0}^{\infty} \mathrm{e}^{-2^n} 2^{(n+1)a}\right) t^{-a}.$$

So (iii) $\Rightarrow$ (ii). $\square$

Theorem 4.1. *Assume* $p_0 = p_1 = 0$ *and* (2.3), *and let* Z *be a non-trivial solution of* (2.1). *Then*

(i) *For all* $a \in (0,\infty)$, $P(Z \leqslant x) = \mathrm{O}(x^a)$ $(x \to 0)$ *if and only if* $P(A \leqslant x) = \mathrm{O}(x^a)$ $(x \to 0)$.

(ii) *If for some constants* $a > 0$, $C_i > 0$ $(i = 1,2)$ *and all* $x > 0$ *small enough*, $C_1 x^a \leqslant P(A \leqslant x) \leqslant C_2 x^a$, *then the same is true for* Z: *there are some constants* $K_i > 0$ $(i = 1,2)$ *such that for all* $x > 0$ *small enough*,

$$K_1 x^a \leqslant P(Z \leqslant x) \leqslant K_2 x^a.$$

(iii) *For all* $a \in [0,\infty)$, *if* $P(A \leqslant x) = \mathrm{O}(x^a)$ $(x \to 0)$, *then*

$$|E\mathrm{e}^{\mathrm{i}tZ}| = \mathrm{O}(|t|^{-a}) \quad (|t| \to \infty);$$

if additionally (2.5) *holds*, *then*

$$\frac{\mathrm{d}E\mathrm{e}^{\mathrm{i}tZ}}{\mathrm{d}t} = \mathrm{O}(|t|^{-(1+a)}) \quad (|t| \to \infty).$$

Proof. Write $m = \min\{i \geqslant 2\colon p_i > 0\}$. Put $\phi(t) = E\mathrm{e}^{-tZ}$ and write $\phi^m(t) = (\phi(t))^m$. We first prove part (i). Assume $P(A \leqslant x) = \mathrm{O}(x^a)$, $x \to 0$. Then by Lemma 4.4, $EA^{-b} < \infty$ for all $0 < b < a$. By Eq. (2.1), $\phi(t) \leqslant E\phi^m(At)$, $t \geqslant 0$. Let $\varepsilon > 0$ be such that $\varepsilon EA^{-b} < 1$. Then

$$\phi^m(t) \leqslant [E\phi^m(At)]^m \leqslant \varepsilon E\phi^m(At).$$

for all $t > 0$ large enough. So by Lemma 4.1, $\phi^m(t) = \mathrm{O}(t^{-b})$, $t \to \infty$. Substituting this in the preceding inequality on $\phi^m(t)$ gives $\phi^m(t) = \mathrm{O}(t^{-mb})$, $t \to \infty$. Take $a/m < b < a$ and let $C > 0$ be such that $\forall t > 0$, $\phi(t) \leqslant C(1+t)^{-b}$. Therefore $\phi(At) \leqslant C(1+At)^{-b}$, and

$$\phi(t) \leqslant E[(1+At)^{-bm}] = \mathrm{O}(t^{-a}),$$

where the last step holds by Lemma 4.3 because $P(A \leqslant x) = \mathrm{O}(x^a)$ $(x \to 0)$ and $bm > a$. Therefore $P(Z \leqslant x) = \mathrm{O}(x^a)$, $x \to 0$. Conversely, if $P(Z \leqslant x) = \mathrm{O}(x^a)$, $x \to 0$, then by Lemma 4.2(ii). $P(A \leqslant x) = \mathrm{O}(x^a)$, $x \to 0$. This ends the proof of part (i). Part (ii) comes directly from part (i) and Lemma 4.2(ii).

For part (iii), let us write $\psi(t) = E\mathrm{e}^{\mathrm{i}tZ}$. The proof of the first conclusion is the same as in the proof of part (i) above, using

$$|\psi(t)| \leqslant E|\psi^m(At)| \quad (t \in \mathbb{R})$$

and the fact that $\lim_{|t| \to \infty} \psi(t) = 0$. For the second conclusion, again by (2.1) (with ϕ replaced by ψ), we have, for all $t \in \mathbb{R}$,

$$\psi'(t) = \sum_{k \geqslant m} kp_k E[(\phi(At))^{k-1}\psi'(At)A]$$

$$|\psi'(t)| \leqslant \sum_{k\geqslant m} kp_k E[|\psi(At)|^{m-1}|\psi'(At)|A]$$
$$\leqslant E[|\psi(Bt)|\,|\psi'(Bt)|],$$

where B is a random variable with distribution $P_B(\mathrm{d}x) = xP_A(\mathrm{d}x)/EA$ (recall that $EAEN = 1$). Let $\varepsilon > 0$ be arbitrary and let $t_\varepsilon > 0$ be such that $|\psi(x)| \leqslant \varepsilon$ if $|x| > t_\varepsilon$. Then

$$E[|\psi(Bt)|\,|\psi'(Bt)|] \leqslant \varepsilon E[|\psi'(Bt)|] + P(|Bt| \leqslant t_\varepsilon). \tag{4.2}$$

By Markov's inequality, it follows that for all $b > -1$ with $EB^{-(1+b)} < \infty$ and for all $t \in \mathbb{R}$,

$$|\psi'(t)| \leqslant \varepsilon E[|\psi'(Bt)|] + EB^{-(1+b)} t_\varepsilon^{1+b} t^{-(1+b)}. \tag{4.3}$$

Therefore, since $EB^{-(1+b)} = EA^{-b}/EA$, by Lemma 4.1 we see that

$$|\phi'(t)| = \mathrm{O}(|t|^{-(1+b)}) \quad (t \to +\infty)$$

whenever $EA^{-b} < \infty$, $b \geqslant 0$. Applying the result for $b = 0$, we see that we have always

$$|\phi'(t)| = \mathrm{O}(|t|^{-1}) \quad (t \to +\infty), \tag{4.4}$$

without any hypothesis on the tail of A. This ends the proof for the case where $a = 0$. Now assume $a > 0$ and take $0 < b < a$. Since $EA^{-b} < \infty$, the preceding conclusion ensures that $|\psi'(t)| = \mathrm{O}(|t|^{-(1+b)})(|t| \to \infty)$. Let $C > 0$ be such that for all $t \in \mathbb{R}, |\psi(t)| \leqslant C(1+|t|)^{-a}$ and $|\psi'(t)| \leqslant C(1+|t|)^{-(1+b)}$. Then

$$|\psi'(t)| \leqslant E[|\psi(Bt)|\,|\psi'(Bt)|]$$
$$\leqslant C^2 E[(1+B|t|)^{-[a+1+b]}]$$
$$= \mathrm{O}(|t|^{a+1}) \quad (|t| \to \infty),$$

where the last step holds by Lemma 4.3 since $P(B \leqslant x) = \mathrm{O}(x^{a+1})\,(x \to 0)$ and $a + 1 + b > a + 1$. This ends the proof of Theorem 4.1. □

5. Left tail and decay rate of $E\mathrm{e}^{\mathrm{i}tZ}$ and its derivative: the case where $p_0 + p_1 > 0$

By the principle of reduction (Proposition 2.1), it suffices to consider the case where $p_0 = 0$.

Theorem 5.1. *Assume* $p_0 = 0, 0 < p_1 < 1$ *and* (2.3), *and let* Z *be a non-trivial solution of* (2.1).

(i) *For all* $0 < a < \infty$, *if* $p_1 EA^{-a} < 1$, *then* $P(Z \leqslant x) = \mathrm{O}(x^a)\,(x \to 0)$; *conversely, if* $EZ^{-a} < \infty$, *then* $p_1 EA^{-a} < 1$.

(ii) *For all* $0 \leqslant a < \infty$, *if* $p_1 EA^{-a} < \infty$, *then*

$$E\mathrm{e}^{\mathrm{i}tZ} = \mathrm{O}(|t|^{-a}) \quad (|t| \to \infty);$$

if additionally (2.5) *holds, then*

$$\frac{\mathrm{d}E\mathrm{e}^{\mathrm{i}tZ}}{\mathrm{d}t} = \mathrm{O}(|t|^{-(1+a)}) \quad (|t| \to \infty).$$

Proof. The proof is similar to that of Theorem 1.1. For part (i), the second assertion is contained in Lemma 4.2; for the first assertion, let us assume $p_1 EA^{-a} < 1$ and write $\phi(t) = E e^{-tZ}$. Then for all $t > 0$,

$$\phi(t) \leqslant p_1 E[\phi(At)] + (1 - p_1) E[\phi^2(At)]. \tag{5.1}$$

Let $\varepsilon > 0$ and $t_\varepsilon > 0$ be such that $(p_1 + \varepsilon) EA^{-a} < 1$ and $\phi(x) \leqslant \varepsilon$ if $x > t_\varepsilon$. Then

$$E[\phi^2(At)] \leqslant \varepsilon E[\phi(At)] + P(At \leqslant t_\varepsilon), \tag{5.2}$$

so that

$$\phi(t) \leqslant (p_1 + \varepsilon) E[\phi(At)] + E(A^{-a}) t_\varepsilon^a t^{-a}.$$

So by Lemma 4.1,

$$\phi(t) = O(t^{-a}) \quad (t \to \infty).$$

Consequently $P(Z \leqslant x) = O(x^a)\,(x \to 0)$ by Lemma 4.4. This ends the proof of part (i). For part (ii), the proof of the first conclusion is the same as above, since (5.1) also holds for $\phi(t) := |\psi(t)|$ with $\psi(t) = E e^{itZ}\,(t \in \mathbb{R})$ and $\lim_{|t| \to \infty} |\psi(t)| = 0$. For the second conclusion, we remark that for all $t \in \mathbb{R}$,

$$\psi'(t) = \sum_{k \geqslant 1} k p_k \psi^{k-1}(At) \psi'(At) A,$$

$$\begin{aligned} |\psi'(t)| &\leqslant p_1 E[|\psi'(At)| A] + \left(\sum_{k > 1} k p_k \right) E[|\psi(At)| |\psi'(At)| A] \\ &= p_1 EAE[|\psi'(Bt)|] + (EN - p_1)(EA) E[|\psi(Bt)| |\psi'(Bt)|], \end{aligned}$$

where B is a random variable with distribution $P_B(dx) = x P_A(dx)/EA$. Remark that $(p_1 EA) E[B^{-(a+1)}] = p_1 EA^{-a} < 1$. Using (4.2) and the same argument as in part (i) above shows that,

$$|\psi'(t)| = O(|t|^{-(a+1)}) \quad \text{as } |t| \to \infty.$$

This is just the result desired. □

Corollary 5.1. *Under the conditions of Theorem* 5.1, *if additionally* $A \leqslant 1$ *almost surely*, *then for all* $a > 0$,

$$E[Z^{-a}] < \infty \textit{ if and only if } p_1 E[A^{-a}] < 1.$$

Proof. By Theorem 5.1, it suffices to prove the if part. Let $a > 0$ be such that $p_1 E[A^{-a}] < 1$. Let Z_i be random variables independent of (A, N) and with the same distribution as Z. By Theorem 5.1, $P(Z \leqslant x) = O(x^a)$, so that $E[(Z_1 + Z_2)^{-a}] < \infty$. Let $\delta > 0$ be arbitrary. By (2.1a),

$$\begin{aligned} &E[Z^{-a} 1\{Z > \delta\}] \\ &\quad = E[(A(Z_1 + \cdots + Z_N))^{-a} 1\{A(Z_1 + \cdots + Z_N) > \delta\}] \\ &\quad \leqslant E[(AZ_1)^{-a} 1\{AZ_1 > \delta\} 1\{N = 1\}] + E[(A(Z_1 + Z_2))^{-a} 1\{N > 1\}] \\ &\quad \leqslant E[(AZ_1)^{-a} 1\{Z_1 > \delta\} 1\{N = 1\}] + E[A^{-a}] E[(Z_1 + Z_2)^{-a}] P\{N > 1\}, \end{aligned}$$

where the last step holds since $A \leqslant 1$ almost surely. Therefore

$$(1 - p_1 E[A^{-a}])E[Z^{-a} 1\{Z > \delta\}] \leqslant (1 - p_1)E[A^{-a}]E[(Z_1 + Z_2)^{-a}].$$

Letting $\delta \to 0$ gives

$$E[Z^{-a}] \leqslant (1 - p_1)E[A^{-a}]E[(Z_1 + Z_2)^{-a}]/(1 - p_1 E[A^{-a}]). \qquad \square$$

One might expect to prove Corollary 5.1 without assuming $A \leqslant 1$ almost surely.

6. Left tail and decay rate of $E\mathrm{e}^{\mathrm{i}tZ}$ the case where ess inf $A > 0$

Theorem 6.1. *Assume* (2.3) *and let Z be a non-trivial solution of* (2.1). *Assume also that either N or A is not almost surely constant, and that* $||N||_{-\infty} := \operatorname{ess\,inf} N \geqslant 2$ *and* $||A||_{-\infty} := \operatorname{ess\,inf} A > 0$. *Write* $\gamma = -\log ||N||_{-\infty} / \log ||A||_{-\infty}$. *Then* $0 < \gamma < 1$ *and the following assertions hold*:

(i) *For some constant* $K_1 > 0$ *and all* $t \geqslant 1$, $E\mathrm{e}^{-tZ} \leqslant \mathrm{e}^{-K_1 t^{\gamma}}$.

(ii) *For all* $\varepsilon > 0$, *there exist some constant* $K_2 > 0$ *such that for all* $t \geqslant 1$, $E\mathrm{e}^{-tZ} \geqslant \mathrm{e}^{-K_2 t^{\gamma+\varepsilon}}$; *the conclusion also holds for* $\varepsilon = 0$ *if* $P(A = ||A||_{-\infty}) > 0$.

(iii) *For some constant* $K_3 > 0$ *and all* $t \in \mathbb{R}$ *with* $|t| \geqslant 1$, $|E\mathrm{e}^{\mathrm{i}tZ}| \leqslant \mathrm{e}^{-K_3 |t|^{\gamma}}$.

Proof. To simplify notations, write $a = ||A||_{-\infty}$ and $m = ||N||_{-\infty} = \min\{i \geqslant 0\colon\ p_i > 0\}$ $(\geqslant 2)$. Clearly $a < 1$ by (2.3). Let $\lambda \in (0,1]$ be defined in (2.3). Remark that $ma \leqslant ma^{\lambda} \leqslant ENE[A^{\lambda}] \leqslant 1$, and the equality holds if and only if $\lambda = 1$ and both N and A are constant a.s. Therefore $0 < \gamma < 1$. Write $\phi(t) = E\mathrm{e}^{-tZ}$. By Eq. (2.1), we have

$$\phi(t) \leqslant [\phi(at)]m \quad \text{for all } t > 0.$$

So for $b = 1/a$ and $t > 0$, $\phi(bt) \leqslant [\phi(t)]^m$ and, by iteration, all $k = 0, 1, \ldots$.

$$\phi(b^k) \leqslant [\phi(1)]^{m^k}.$$

Since $m = a^{-\gamma} = b^{\gamma}$, this shows that, for all $k = 0, 1, \ldots,$

$$-\log \phi(b^k) \geqslant K(b^k)^{\gamma}, \quad \text{where } K = -\log \phi(1) > 0.$$

For all $t \geqslant 1$, let $k = k(t) \in \{0, 1, \ldots\}$ be such that $b^k \leqslant t < b^{k+1}$. Then by the monotonicity of ϕ,

$$-\log \phi(t) \geqslant -\log \phi(b^k) \geqslant K(b^k)^{\gamma} \geqslant Ka^{\gamma}(b^{k+1})^{\gamma} \geqslant Ka^{\gamma} t^{\gamma}.$$

This ends the proof of part (i). Part (iii) can be shown in a similar way by considering $\bar{\phi}(t) = \sup_{|s| \geqslant t} |E\mathrm{e}^{\mathrm{i}sZ}|$ instead of $\phi(t)$.

We now come to the proof of part (ii). Let $\varepsilon > 0$ be such that $a + \varepsilon < 1$. Again by Eq. (2.1), we have

$$\begin{aligned} \phi(t) &\geqslant p_m E[\phi^m(At)]) \\ &\geqslant p_m E[\phi^m(At) 1\{A \leqslant a + \varepsilon\}] \\ &\geqslant p_m P[A \leqslant a + \varepsilon] \phi^m((a + \varepsilon)t). \end{aligned}$$

Therefore, for $b = b_{\varepsilon} := 1/(a + \varepsilon) > 1$, $p = p_{\varepsilon} := p_m P[A \leqslant a + \varepsilon] \in (0, 1)$ and all $t > 0$,

$$\phi(bt) \geqslant p[\phi(t)]^m.$$

Iterating, we obtain

$$\phi(b^k) \geqslant [\phi(1)]^{m^k} p^{\sum_{1\leqslant i\leqslant k} m^i}, \quad k = 0, 1, \ldots$$

(where the empty sum is taken to be 0). It follows that for all $k = 0, 1, \ldots,$

$$-\log \phi(b^k) \leqslant m^k \left[-\log \phi(1) + (-\log p) m^{-k} \sum_{1\leqslant i\leqslant k} m^i \right]$$

$$= m^k[-\log \phi(1) + (-\log p) m^{-k} m(m^k - 1)/(m-1)]$$

$$\leqslant (b^k)^{\gamma_\varepsilon} K_\varepsilon,$$

where $\gamma_\varepsilon := \log m / \log 1/(a+\varepsilon)$ and $K_\varepsilon := -\log \phi(1) + (-\log p) m/(m-1)$. As in the proof of part (i), together with the monotony of ϕ, this implies that, for all $t \geqslant 1$,

$$-\log \phi(t) \leqslant K_\varepsilon a^{-\gamma_\varepsilon} t^{\gamma_\varepsilon}.$$

As $\varepsilon > 0$ is arbitrary, this gives the first assertion in part (ii) of the theorem. If additionally $P(A = a) > 0$, it is easily seen that the preceding argument also holds for $\varepsilon = 0$, giving the second assertion in part (ii). So the proof is finished. □

In the case where $A = a$ is constant, Theorem 6.1 is due to Harris (1948). The following results are consequence of Theorem 6.1, using Tauberian theorems of exponential type – see Kasahara (1978) and Liu (1996).

Corollary 6.1. *Under the conditions of Theorem* 6.1, *we have*:

(i) *For all* $\varepsilon > 0$, *there exist some constants* $C_i > 0$ $(i = 1, 2)$ *such that for all* $x > 0$ *small enough*,

$$\exp\{-C_1 x^{-(\gamma/(1-\gamma)+\varepsilon)}\} \leqslant P(Z \leqslant x) \leqslant \exp\{-C_2 x^{-(\gamma/(1-\gamma))}\}.$$

(ii) *Write* $||Z^{-1}||_k = (E[Z^{-k}])^{1/k}$. *For all* $\varepsilon > 0$, *there exist some constants* $C_i > 0$ $(i = 3, 4)$ *such that for all* $k \geqslant 1$,

$$C_3 k^{(1-\gamma)/\gamma - \varepsilon} \leqslant ||Z^{-1}||_k \leqslant C_4 k^{(1-\gamma)/\gamma}.$$

(iii) $E \exp(Z^{-q}) < \infty$ if $q < \gamma/(1-\gamma)$, *and* $E \exp(Z^{-q}) = \infty$ if $q > \gamma/(1-\gamma)$.

(iv) *If* $\gamma > \frac{1}{2}$, *then* Z^{-1} *has an entire characteristic function of order* $\gamma/(2\gamma - 1) > 1$, *and, for all* $\varepsilon > 0$, *there are some constants* $C_5 > 0$ *and* $C_6 > 0$ *such that, for all* $t > 0$ *large enough*,

$$C_5 t^{\gamma/(2\gamma-1)-\varepsilon} \leqslant \log E \mathrm{e}^{tZ^{-1}} \leqslant C_6 t^{\gamma/(2\gamma-1)}.$$

(v) *If* $P(A = a) > 0$, *then all the conclusions in* (i), (ii) *and* (iv) *also hold for* $\varepsilon = 0$, *and the assertion* (iii) *can be improved to the following*: *for some but not all* $r > 0$,

$$E \exp(rZ^{-\gamma/(1-\gamma)}) < \infty.$$

7. Moments and polynomial right tail

We only consider the case where (2.5) holds. The contrary case was considered in Liu (1998).

Theorem 7.1. *Assume* (2.5) *and let* Z *be a nontrivial solution of* (2.1). *Then for each fixed* $p > 1$, $E(Z^p) < \infty$ *if and only if*

$$E[N^p] < \infty \quad \textit{and} \quad ENE[A^p] < 1.$$

Notice that if $||A||_p \leqslant 1$ for all $p > 1$, then $||A||_\infty \leqslant 1$. So by Theorem 7.1, we have:

Corollary 7.1. *Assume* (2.5) *and let* Z *be a nontrivial solution of* (2.1). *Then* Z *has moments of all orders if and only if so does* N, *together with* $A \leqslant 1$ *almost surely.*

Proof of Theorem 7.1. For simplicity, we assume $EZ = 1$. By Eq. (2.1),

$$\phi'(t) = E[f'(\phi(At))\phi'(At)A], \quad t \in \mathbb{R}, \tag{7.1}$$

where $\phi(t) = E\mathrm{e}^{\mathrm{i}tZ}$. Let $\tilde{Z}$ be a real random variable with distribution $P_{\tilde{Z}}(\mathrm{d}x) = xP_Z(\mathrm{d}x)$, and let $(\tilde{A}, \tilde{B})$ be a $\mathbb{R}^2$-valued random variable independent of $\tilde{Z}$, whose distribution is determined by

$$Eh(\tilde{A}, \tilde{B}) = \sum_{n=1}^{\infty} n p_n E\left[Ah\left(A, A\sum_{i=1}^{n-1} Z_i\right)\right]$$

(the empty sum is always taken to be 0), where h is an arbitrary bounded and measurable function on $\mathbb{R}^2$. Then $\tilde{Z}$ has characteristic function $\tilde{\phi}(t) = \phi'(t)/i$, and Eq. (7.1) reads

$$\tilde{\phi}(t) = E[\mathrm{e}^{\mathrm{i}t\tilde{B}}\tilde{\phi}(\tilde{A}t)], \quad t \in \mathbb{R},$$

namely,

$$\tilde{Z} \stackrel{\mathrm{d}}{=} \tilde{A}\tilde{Z} + \tilde{B}. \tag{7.1a}$$

Notice that by the definition of $(\tilde{A}, \tilde{B})$, we have

$$P(\tilde{A} > 0) = P(\tilde{B} \geqslant 0) = 1, \quad E\log\tilde{A} = ENEA\log A < 0,$$

$$E\tilde{A}^{p-1} = ENEA^p \quad \text{and} \quad E\tilde{B}^{p-1} = EA^pE\left[N\left(\sum_{i=1}^{N-1} Z_i\right)^{p-1}\right].$$

Let $\{(\tilde{A}_n, \tilde{B}_n)\colon n \geqslant 1\}$ be independent copies of $(\tilde{A}, \tilde{B})$, then

$$\tilde{Z} \stackrel{\mathrm{d}}{=} \tilde{B}_1 + \tilde{A}_1\tilde{B}_2 + \tilde{A}_1\tilde{A}_2\tilde{B}_3 + \cdots$$

so that by the triangular inequality in L^{p-1},

$$||\tilde{Z}||_{p-1} \leqslant ||\tilde{B}||_{p-1}/(1 - ||\tilde{A}||_{p-1}) \quad \text{whenever } ||A||_{p-1} < 1, \quad p > 1. \tag{7.2}$$

Now if $p - 1 \leqslant 1$, then by the concaveness of the function $x \mapsto x^{p-1}$,

$$E\left[\left(\sum_{i=1}^{N-1} Z_i\right)^{p-1} |N\right] \leqslant \left[E\left(\sum_{i=1}^{N-1} Z_i\right) |N\right]^{p-1} = (N-1)^{p-1},$$

so that

$$E\left[N\left(\sum_{i=1}^{N-1} Z_i\right)^{p-1}\right] = E\left[NE\left[\left(\sum_{i=1}^{N-1} Z_i\right)^{p-1} \,|\, N\right]\right]$$
$$\leqslant E[N(N-1)^{p-1}] \leqslant E[N^p];$$

if $p-1>1$, then by the convexity of the function $x \mapsto x^{p-1}$,

$$\left(\frac{1}{N-1}\sum_{i=1}^{N-1} Z_i\right)^{p-1} \leqslant \frac{1}{N-1}\sum_{i=1}^{N-1} Z_i^{p-1} \quad \text{if } N>1,$$

so that

$$E\left[N\left(\sum_{i=1}^{N-1} Z_i\right)^{p-1}\right] \leqslant E[N^p]E[Z^{p-1}].$$

Therefore,

$$E[\tilde{B}^{p-1}] \leqslant \begin{matrix} E[N^p]E[A^p] & \text{if } p-1 \leqslant 1, \\ E[N^p]E[A^p]EZ^{p-1} & \text{if } p-1>1. \end{matrix} \tag{7.3}$$

Since $E[\tilde{Z}^{p-1}] = E[Z^p]$, using (7.2),(7.3) and an induction argument on n, we see that for all $n \geqslant 2$, if $p \in (n-1, n]$, $EN^p < \infty$ and $ENEA^p < 1$, then $E[Z^p] < \infty$. This gives the sufficiency of the theorem.

Now assume $EZ^p < \infty$. Then by Eq. (2.1a),

$$\begin{aligned} E[Z^p] &= E[A^p]E[(Z_1 + \cdots + Z_N)^p] \\ &> E[A^p]E[Z_1^p + \cdots + Z_N^p] = ENE[A^p]E[Z^p], \end{aligned}$$

so that $ENE[A^p] < 1$; on the other hand, by Jensen's inequality,

$$\begin{aligned} E[Z^p] &= E[A^p]E[E[(Z_1 + \cdots + Z_N)^p \,|\, N]] \\ &\geqslant E[A^p]E[E[(Z_1 + \cdots + Z_N) \,|\, N]^p] \\ &= E[A^p]E[N^p], \end{aligned}$$

so that $E[N^p] < \infty$. This ends the proof of Theorem 7.1. □

Theorem 7.2. *Assume* (2.5) *and let* Z *be a nontrivial solution of* (2.1). *Suppose that for some* $\chi > 1$,

$$ENE[A^\chi] = 1, \quad E[A^\chi \log^+ A] < \infty \quad \text{and} \quad E[N^\chi] < \infty.$$

If the problem is non-lattice, then there is a constant $c \in (0, \infty)$ *such that*

$$\lim_{x\to\infty} x^\chi P(Z > x) = c;$$

if the problem is lattice, then

$$0 < \liminf_{x\to+\infty} x^\chi P(Z > x) \leqslant \limsup_{x\to+\infty} x^\chi P(Z > x) < \infty.$$

Proof. We can assume again $EZ = 1$. By Theorem 7.1 and its proof, under the given conditions we have

$$E(\tilde{A}^{\chi-1}) = 1, \quad E(\tilde{A}^{\chi-1}\log^+ \tilde{A}) < \infty \quad \text{and} \quad E[\tilde{B}^{\chi-1}] < \infty. \tag{7.4}$$

If the problem is non-lattice, then by a theorem of Kesten (1973) (see also Grintsevichyus, 1975) about the random difference equation (7.1a), (7.4) implies that the limit

$$\lim_{t\to+\infty} t^{\chi-1}\int_t^\infty xP_Z(\mathrm{d}x) = \lim_{t\to+\infty} t^{\chi-1}P(\tilde{Z} > t)$$

exists and is strictly positive and finite. By Chapter VIII.9, Theorem 2 of Feller (1970), this implies that the limit

$$\lim_{t\to+\infty} t^{\chi}P(Z > t)$$

exists and is strictly positive and finite. If the problem is lattice with span $h > 0$, then by a theorem of Grintsevichyus (1975) about the same Eq. (7.1a), the condition (7.4) implies that for all real x,

$$\lim_{n\to\infty} \mathrm{e}^{(x+nh)\chi-1}P(\tilde{Z} > \mathrm{e}^{x+nh}(\chi-1)) = d(x),$$

where $d(x) \in (0,\infty)$ is a strictly positive and h-periodic function on $\mathbb{R}$. In particular,

$$0 < \liminf_{t\to+\infty} t^{\chi-1}P(\tilde{Z} > t) \leqslant \limsup_{t\to+\infty} t^{\chi-1}P(\tilde{Z} > t) < \infty.$$

This gives

$$0 < \liminf_{t\to+\infty} t^{\chi}P(Z > t) \leqslant \limsup_{t\to+\infty} t^{\chi}P(Z > t) < \infty. \qquad \square$$

The arguments of this section can be extended to more general cases: see Liu (1997b).

8. Exponential moments and analyticity of characteristic function; exponential right tail and growth order of entire characteristic function

Theorem 8.1. *Assume* (2.3) *and let Z be a non-trivial solution of* (2.1). *Then the following assertions are equivalent*: (i) $A \leqslant 1$ *almost surely*, $ENEA = 1$ *and* $E\mathrm{e}^{tNA} < \infty$ *for some* $t > 0$; (ii) *for some* $t > 0$, $E\mathrm{e}^{tZ} < \infty$.

Notice that a real random variable X has an analytic characteristic function if and only if $E\mathrm{e}^{t|X|} < \infty$ for some $t > 0$, we obtain immediately:

Corollary 8.1. *Z has an analytic characteristic function if and only if so does N, together with* $P(A \leqslant 1) = 1$ *and* $ENEA = 1$.

Proof of Theorem 8.1. (ii) $\Rightarrow$ (i). Assume (ii). Let $\sigma(N, A)$ be the σ-field generated by (N, A). By Jensen's inequality, we have, for all $t \in \mathbb{R}$,

$$E\mathrm{e}^{tZ} = E\mathrm{e}^{tA(Z_1+\cdots+Z_N)}$$

$$
\begin{aligned}
&= E[E[\mathrm{e}^{tA(Z_1+\cdots+Z_N)} \mid \sigma(N,A)]] \\
&\geqslant E[\mathrm{e}^{E[tA(Z_1+\cdots+Z_N) \mid \sigma(N,A)]}] \\
&= E\mathrm{e}^{tNA}.
\end{aligned}
$$

Therefore for all $t>0$, $E\mathrm{e}^{tZ}<\infty$ implies $E\mathrm{e}^{tNA}<\infty$. On the other hand, (ii) implies in particular that Z has finite moments of all orders, so that (2.5) holds by Proposition 2.2(ii), and $A\leqslant 1$ almost surely by Corollary 7.1.

(i) $\Rightarrow$ (ii) This can be considered as a special case of a theorem due to Rösler (1992). For convenience of readers and for the paper to be self-contained, we present a proof, following Rösler (1992). Assume (i). Let $\{N_u\colon u\in\mathbb{U}\}$ and $\{A_u\colon u\in\mathbb{U}\}$ be two independent families of independent random variables, with N_u having the same distribution as N and A_u the same as A. Let $\mathbb{T}=\mathbb{T}(\omega)$ be the corresponding Galton–Watson tree associated with $\{N_u\colon u\in\mathbb{U}\}$: we have $\emptyset\in\mathbb{T}$ and, if $u\in\mathbb{T}$ and $i\in\mathbb{N}$, then $ui\in\mathbb{T}$ if and only if $1\leqslant i\leqslant N_u$. Put

$$Y_0=1 \quad \text{and} \quad Y_{n+1}=\sum_{u=u_1\ldots u_n u_{n+1}\in\mathbb{T}} A_\emptyset A_{u_1}\ldots A_{u_1\ldots u_n}, \quad n\geqslant 0.$$

Then $\{Y_n\colon n\geqslant 0\}$ is a martingale and $Z\overset{\mathrm{d}}{=}\lim_{n\to\infty}Y_n$. Write $\phi_n(t)=E\mathrm{e}^{tY_n}$, $n\geqslant 0$. Consider the function

$$g_K(t)=E\mathrm{e}^{(AN-1)t+K(NA^2-1)t^2}, \quad t\geqslant 0,$$

where $K>0$ is a constant to be chosen later. We have $g_K(0)=1$, $g_K'(0)=0$ and $g_K''(0)=E[(AN-1)^2]+2KE[NA^2-1]$. Let $K>0$ be sufficient large such that $g_K''(0)<0$. This is possible because $E[NA^2-1]<0$. Let $t_K>0$ be small enough such that $g_K''(t)<0$ for all $t\in[0,t_K]$. Therefore $g_K(t)\leqslant g_K(0)=1$ if $0\leqslant t\leqslant t_K$. Consequently, if $\phi_n(t)\leqslant \mathrm{e}^{t+Kt^2}$ for all $0\leqslant t\leqslant t_K$, then for all these t,

$$
\begin{aligned}
\phi_{n+1}(t) &= E[(\phi_n(At))^N] \\
&\leqslant E[(\mathrm{e}^{At+K(At)^2})^N] \quad (\text{since } At\leqslant t\leqslant t_K) \\
&= \mathrm{e}^{t+Kt^2} g_K(t) \\
&\leq \mathrm{e}^{t+Kt^2}.
\end{aligned}
$$

Notice that $\phi_0(t)=\mathrm{e}^t$. So by induction on n, we have proved that for all $n\geqslant 0$ and all $t\in[0,t_K]$, $E\mathrm{e}^{tY_n}\mathrm{e}\leqslant \mathrm{e}^{t+Kt^2}$. By Fatou's lemma, this gives $E\mathrm{e}^{tZ}\leqslant\mathrm{e}^{t+Kt^2}$ if $0\leqslant t\leqslant t_K$. So the proof is finished. $\square$

A similar argument as in the proof of Theorem 6.1 yields the following:

Theorem 8.2. *Assume* (2.3) *and let Z be a non-trivial solution of* (2.1). *Assume also that either N or A is not almost surely constant, and that $\|N\|_\infty := \operatorname{ess\,sup} N<\infty$ and $\|A\|_\infty := \operatorname{ess\,sup} A<1$. Write $\bar\gamma=-\log\|N\|_\infty/\log\|A\|_\infty$. Then $\bar\gamma>1$ and the following assertions hold*:

(i) *For some constant $K_1>0$ and all $t\geqslant 1$, $E\mathrm{e}^{tZ}\leqslant \mathrm{e}^{K_1 t^{\bar\gamma}}$.*

(ii) *For all $\varepsilon>0$, there exist some constant $K_2>0$ such that for all $t\geqslant 1$, $E\mathrm{e}^{tZ}\geqslant \mathrm{e}^{K_2 t^{\bar\gamma+\varepsilon}}$; the conclusion also holds for $\varepsilon=0$ if $P(A=\|A\|_\infty)>0$.*

The following result is similar to Corollary 6.1. It also follows from Theorem 8.2 by Tauberian theorems of exponential type.

Corollary 8.2. *Under the conditions of Theorem* 8.2, *we have*:

(i) *For all* $\varepsilon > 0$, *there exist some constants* $C_i > 0$ ($i=1,2$) *such that for all* $x > 0$ *small enough*,

$$\exp\{-C_1 x^{(\bar\gamma/(\bar\gamma-1))+\varepsilon)}\} \leqslant P(Z > x) \leqslant \exp\{-C_2 x^{(\bar\gamma/(\bar\gamma-1))}\}.$$

(ii) *Write* $||Z||_k = (E[Z^k])^{1/k}$. *For all* $\varepsilon > 0$, *there exist some constants* $C_i > 0$ ($i = 3,4$) *such that for all* $k \geqslant 1$,

$$C_3 k^{(\bar\gamma-1)/\bar\gamma-\varepsilon} \leqslant ||Z||_k \leqslant C_4 k^{(\bar\gamma-1)/\bar\gamma};$$

Consequently, $\lim_{k\to\infty} \log||Z||_k/\log k = (\bar\gamma-1)/\bar\gamma$.

(iii) $E\exp(Z^q) < \infty$ if $q < \bar\gamma/(\bar\gamma-1)$, *and* $E\exp(Z^q) = \infty$ if $q > \bar\gamma/(\bar\gamma-1)$.

(iv) Z *has an entire characteristic function of regular growth with order* $\bar\gamma$ *and with finite type.*

(v) *If* $P(A=||A||_\infty) > 0$, *then*: (a) *all the conclusions in* (i), (ii) *also hold for* $\varepsilon=0$, (b) *the assertion* (iii) *can be improved to the following*: *for some but not all* $r > 0$, $E\exp(rZ^{\bar\gamma/(\bar\gamma-1)}) < \infty$; (c) Z *has an entire characteristic function with order* $\bar\gamma$ *and with finite type and non-zero lower type.*

9. Remarks and questions

(i) One could expect to improve Corollary 1.2 to the following: there exist some constants $C_i > 0$ such that for all $x > 0$ small enough and all $t \in \mathbb{R}$ with $|t|$ large enough,

$$C_1 x^\beta \leqslant P(0 < Z \leqslant x) \leqslant C_2 x^\beta$$

and

$$C_3 |t|^{-\beta} \leqslant |E(\mathrm{e}^{\mathrm{i}tZ} \mid Z > 0)| \leqslant C_4 |t|^{-\beta}.$$

This is true for the Galton–Watson process – the case where $L = 1$ almost surely, see Bingham (1988). In the non-lattice case, that is, if there does not exist $h > 0$ such that $P(L \in h\mathbb{Z}) = 1$, where $h\mathbb{Z} = \{0, \pm h, \pm 2h, \ldots\}$, one might also expect to prove that the limits

$$\lim_{x\to 0} P(0 < Z \leqslant x)/x^\beta \quad \text{and} \quad \lim_{|t|\to\infty} |E(\mathrm{e}^{\mathrm{i}tZ} \mid Z > 0)||t|^\beta$$

exist with values strictly positive and finite.

(i) In Theorem 1.2(ii), if the condition is replaced by $P(A \leqslant x) \sim x^a l(x) (x \to 0)$ for some function l slowly varying at 0, then we can prove that $P(Z \leqslant x) \sim Cx^a l(x) (x \to 0)$ for some constant $0 \leqslant C < \infty$: this has recently been shown in Liu (1999a).

(ii) In Theorem 1.4, can one remove the condition $P(A = ||A||_{-\infty}) > 0$? If not, it would be interesting to relax this condition. It is also interesting to know whether the limits

$$\lim_{x\to 0} -x^{\gamma/(1-\gamma)} \log P(Z \leqslant x) \quad \text{and} \quad \lim_{|t|\to\infty} -|t|^{-\gamma} \log |E e^{itZ}|$$

exist.

(iii) Assuming only (2.2) and (2.3) and let Z be a non-trivial solution of (2.1). Is the distribution of Z always absolutely continuous on $(0,\infty)$? By Theorems 4.1, 5.1 and Proposition 2.1, this is the case if one of the following conditions holds: (a) $p_0 = p_1 = 0$ and $EA^{-a} < \infty$ for some $a > \frac{1}{2}$; (b) $p_0 + p_1 > 0$ and $f'(q)EA^{-a} < 1$ for some $a > \frac{1}{2}$; (c) (2.5).

(iv) In the model of branching random walk, if we define $S_u = L_{u_1} + \cdots + L_{u_1 \ldots u_n}$, $u = u_1 \ldots u_n$, then the Laplace transform ϕ of the corresponding limit variable satisfies the equation

$$\phi(t) = f(E[\phi(At)]). \tag{9.1}$$

In fact, this time instead of (0.5) we have

$$Z = A_1 Z_1 + \cdots + A_N Z_N \tag{9.1a}$$

with $A_i = e^{-\alpha L_i}/m(\alpha)$ independent of each other, Z_i being independent copies of Z which are also independent of $(N, A_1, A_2, \ldots)$; this gives (9.1). The two Eqs. (2.1) and (9.1) have similar properties; Eq. (9.1) contains Mandelbrot's functional equation arising in self-similar cascades, and is studied in the sequel Liu (1999a). For a more general setting, see Liu (1999b).

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